REAL-TIME ALGORITHMS FOR DIGITAL TWINS

Reduced Order Modelling

UKACM Autumn School 2026
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1 Reduced Order Modelling – Introduction

2 Mode Displacement Method

3 Krylov Subspace Model Order Reduction

4 Reduced Order Modelling – Wrap Up

Reduced Order Modelling
Modelling and Simulation

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Task [Continuous Formulation]:

Find u such that

$$\partial_t u = \Delta u + \vec{b} \cdot \vec{c} \text{ in } \Omega \text{ plus appropriate boundary conditions on } \partial\Omega.$$

Task [Discrete Variational Formulation]:

Find $u_h = \sum_{i=1}^{N} x_i(t)\, \phi_h^i(x)$ such that

$$\boldsymbol{M} \cdot \partial_t \vec{x} = \boldsymbol{A} \cdot \vec{x} + \boldsymbol{B} \cdot \vec{c}$$

with $M_{ij} = \int_\Omega \phi_h^{(j)} \phi_h^{(i)} dx$, $A_{ij} = \int_\Omega \nabla\phi_h^{(j)} \cdot \nabla\phi_h^{(i)} dx$, and $B_{ij} = \int_\Omega b_j \phi_h^{(i)} dx$.

Challenge:

Depending on the problem, the Finite Element Ansatz Spaces might be rather large, depending on the problem up to $N \approx 10^9$. Thus, we have to solve (non-)linear equations of that size.

Reduced Order Modelling
Core Idea – Projection-based Reduced Order Modelling (i)

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Idea [Model Order Reduction]:

The goal of Model Order Reduction is

- *Reduce the size of the variables* in the solution (reduction of states)
- *Reduce the number of model equations* (reduction of dynamics)
- While keeping the crucial aspects of the model equations and the solution

History [Model Order Reduction]:

- Model Order Reduction methods have been pursued in-dependently in different fields, such as Mechanical Analysis (e.g. Nodal Displacement Methods), (Mathematical) Numerical Analysis (e.g., Krylov Subspace Methods), or the Systems and Control Community (e.g., Balanced Truncation Methods).
- Today, we have a broad toolbox of different Model Order Reduction technologies, but since it is a very active research field the number of methods is growing continuously.

Taking a Finite Element view for the moment:

Is there an appropriate Ansatz space $\phi_h^i(x)$ with a „minimal“ N ?

Reduced Order Modelling

Core Idea – Projection Based Reduced Order Modelling (ii)

► Let us assume, the we have found a great function space with minimal $\boldsymbol{N}$. We can then represent these functions always in a classical finite element representation, e.g., a piecewise linear formulation. I.e., we can represent the bases of the different function spaces with respect to each other.

Orthogonal Projection:

Let v_1, ..., v_k be an orthonormal basis of ROM (formulated in terms of FOM coordinates) and define the matrix $\boldsymbol{V}$ column-wise by

$$\boldsymbol{V} \coloneqq (v_1, \dots, v_k) \in \mathbb{R}^{N \times K}.$$

This is an orthogonal projection onto the ROM by

$$\boldsymbol{V} \cdot \boldsymbol{V}^T: FOM \to ROM$$

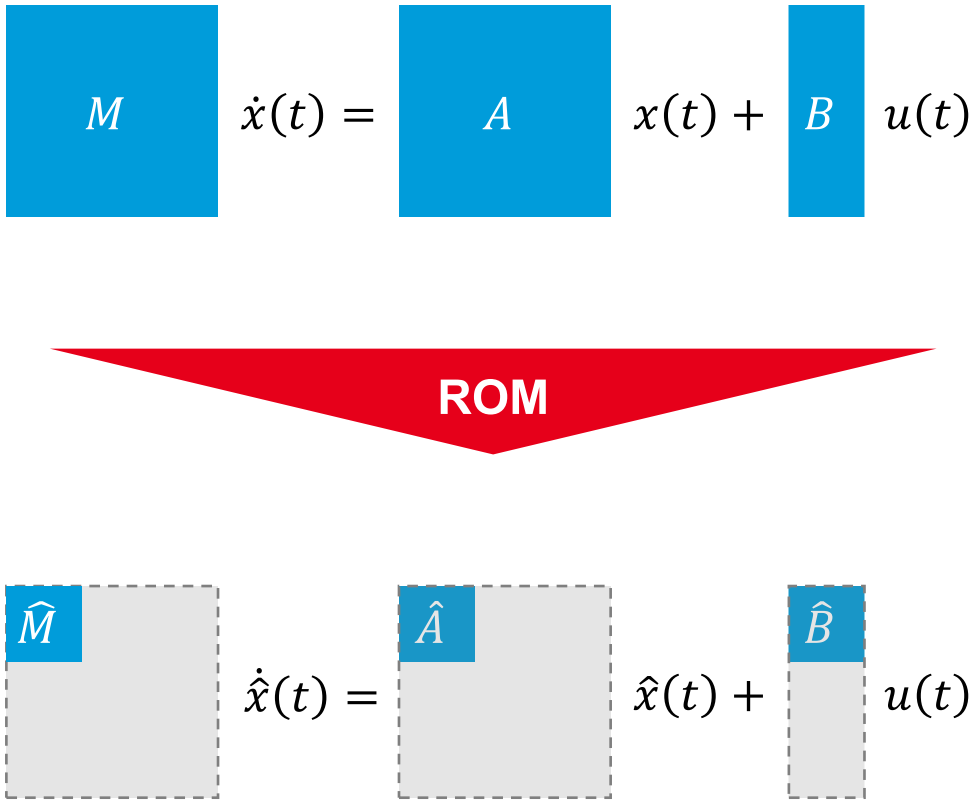
with $\boldsymbol{V} \cdot \boldsymbol{V}^T = \boldsymbol{I}$ on ROM.

► We then can formulate the reduced model by a simple projection as follows

$$\begin{array}{c} \vec{x} \in \mathbb{R}^N \\ \mathbf{M}, \mathbf{A} \in \mathbb{R}^{N \times N} \\ \mathbf{B} \in \mathbb{R}^{N \times M} \end{array} \xrightarrow[\boldsymbol{V}^{\boldsymbol{T}} \in \mathbb{R}^{r \times N}]{\boldsymbol{V} \in \mathbb{R}^{N \times r}} \begin{array}{r} \vec{x}_r \in \mathbb{R}^r \\ \boldsymbol{M}_r = \boldsymbol{V}^T \cdot \boldsymbol{M} \cdot \boldsymbol{V}, \boldsymbol{A}_r = \boldsymbol{V}^T \cdot \boldsymbol{A} \cdot \boldsymbol{V} \in \mathbb{R}^{r \times r} \\ \boldsymbol{B}_r = \boldsymbol{V}^T \cdot \mathbf{B} \in \mathbb{R}^{r \times M} \end{array}$$

Reduced Order Modelling

Core Idea – Projection Based Reduced Order Modelling (iii)



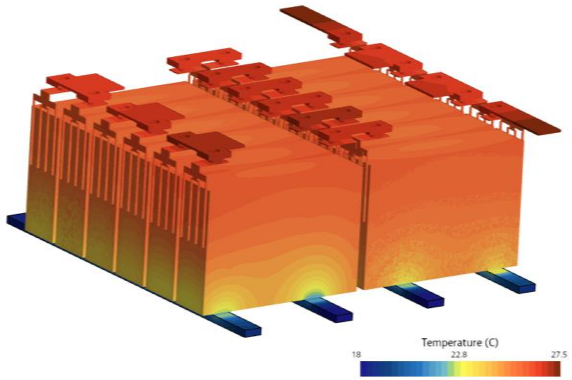
$\mathbf{M}, \mathbf{A} \in \mathbb{R}^{N \times N}$
$\mathbf{B} \in \mathbb{R}^{N \times M}$

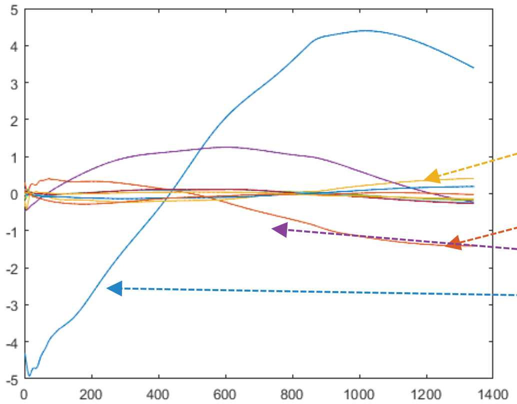
$\mathbf{M}_\mathrm{r}, \mathbf{A}_\mathrm{r} \in \mathbb{R}^{r \times r}$
$\mathbf{B}_r \in \mathbb{R}^{r \times M}$

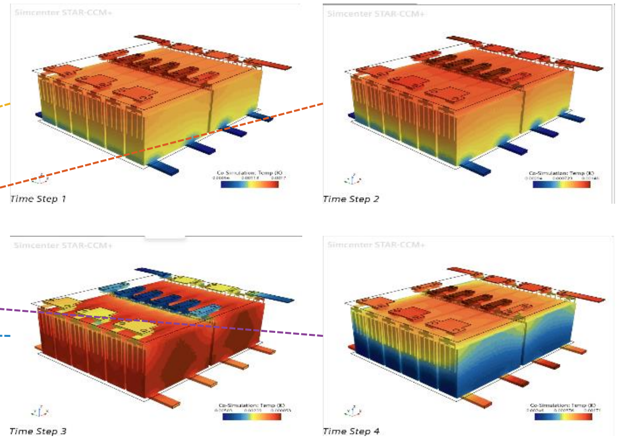
Reduced Order Modelling
Core Idea

$$u_h = \sum_{i=1}^{N} x_i(t)\, \phi_h^i(x)$$

Full Order Model = **Reduced Model** X **Latent / Spectral Modes**



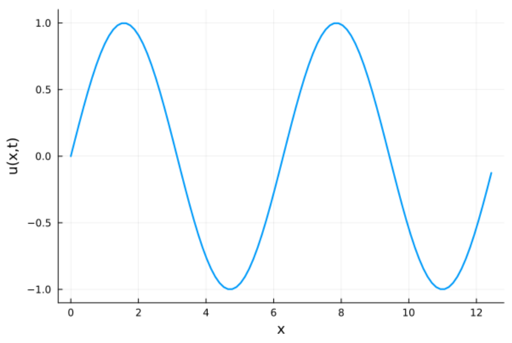




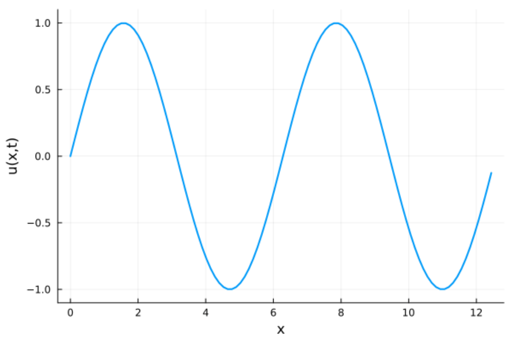
Reduced Order Modelling
Core Idea – Projection Based Reduced Order Modelling (iii)

- Finding a good low dimensional basis is the main challenge (c.f., Spectral Finite Elements). Good bases can be very problem dependent.

Example 1:

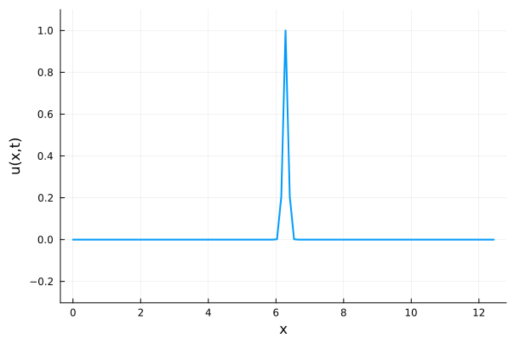


Finite Differences

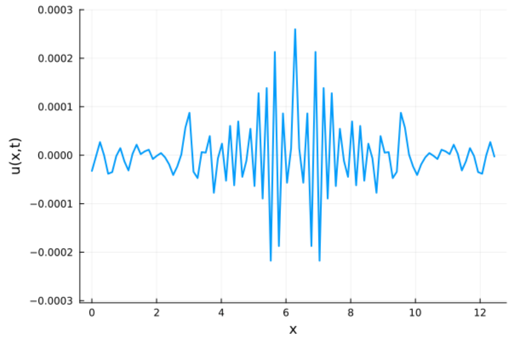


2 modes

Example 2:



Finite Differences



2 modes

Reduced Order Modelling
Overview of Methods

Full Order Model = **Reduced Model** X **Latent / Spectral Modes**

Reduced Model

- Krylov Methods
- Proper Orthogonal Decomposition (POD)
- Autoencoder
- Diffusion Maps
- …

Latent / Spectral Modes

- Galerkin Projection
- Gappy POD
- Discrete Empirical Interpolation Method
- Sparse identification of nonlinear dynamics
- Operator Inference
- Neural Networks
- …

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Reduced Order Modelling

Mode Displacement Method (i) [Structural Analysis community]

- The Mode Displacement Method has evolved in the structural analysis community, considering discretized (e.g., using the finite element method) structural systems of the form:

$$\boldsymbol{M} \cdot \partial_{tt}\vec{u} + \boldsymbol{K} \cdot \vec{u} = \vec{f}$$

- In structural analysis one is often interested in eigenmodes, which are calculated via the *Generalized Eigenvalue System*:

$$\left(\mathrm{K} - \omega_j^2 M\right) \cdot \vec{\phi}_j = 0$$

Idea [Mode Displacement Reduction]:

represent the system in eigen / vibration modes i.e. $\vec{u}(x,t) = \sum_{j=1}^{N} \vec{\psi}_j(x)\eta_j(t)$ (Vibration Mode superposition)

- Expansion along modes, i.e., $\vec{\mathrm{q}} = \sum_{j=1}^{N} \vec{\phi}_j(x)\eta_j(t)$ yields

$$\vec{\phi}_i^T \cdot M \cdot \vec{\phi}_j = \delta_{ij}$$
$$\vec{\phi}_i^T \cdot K \cdot \vec{\phi}_j^T = \delta_{ij}\omega_j^2$$

Respectively

$$\partial_{tt}\eta_j + \omega_j^2\eta_j = \vec{\phi}_j^T \cdot \vec{f} \text{ for } 1 \leq j \leq N$$

Reduced Order Modelling

Mode Displacement Method (ii) [Structural Analysis community]

- Since computation of all eigenvalues is time consuming and analysts are usually interested in the response of the system to lower frequency ranges, thus typically the lowest modes are chosen:

$$\vec{u} = \sum_{j=1}^{K} \vec{\phi}_j(x)\eta_j(t) + \underbrace{\sum_{k=K+1}^{N} \vec{\phi}_j(x)\eta_j(t)}_{\text{Truncated}}$$

- The first modes can be computed with efficient algorithms such as the Arnoldi or Lanczos method
- Projecting to a subsystem with the projection matrix $\boldsymbol{\Phi} = [\vec{\phi}_1, \dots, \vec{\phi}_K]$, we thus obtain

$$\boldsymbol{M_r} \cdot \partial_{tt}\, \vec{\eta} + \boldsymbol{K}_r \cdot \vec{\eta} = \vec{f}_r$$

with $\boldsymbol{M}_r = \boldsymbol{\Phi}^T \cdot \boldsymbol{M} \cdot \boldsymbol{\Phi} = \boldsymbol{I}$, $\boldsymbol{K}_r = \boldsymbol{\Phi}^T \cdot \boldsymbol{K} \cdot \boldsymbol{\Phi} = \mathrm{diag}[\omega_1^2, \dots, \omega_k^2]$, and $\vec{f}_r = \boldsymbol{\Phi}^T \cdot \vec{f}$

Attention:

It is important that …

- … the chosen eigenfrequencies are close to the excitation frequency
- … the excitation is aligned with the chosen eigenmodes, otherwise $\Phi^T, \vec{f}_r$ is close to zero

Reduced Order Modelling

Mode Displacement Method (iii) [Structural Analysis community]

- Mode Displacement is a physically inspired model order reduction technique
- Effective computational methods for identification of the first eigenmodes are available (e.g., the Arnoldi and Lanczos algorithm)

Challenges:

- The specific loading information is ignored, e.g., the selected eigenvectors might be nearly orthogonal to the applied loading and would not participate significantly in the solution
- Requires additional extension to include omitted modes (e.g., static correction method, the mode acceleration method, or the modal truncation method)
- The number of required eigenmodes for a sufficient accuracy is difficult to estimate a priori (limiting the automatic selection of eigenmodes)
- Computation of eigenvectors for large systems is expensive and time consuming

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Reduced Order Modelling
Krylov Subspace Model Order Reduction (i) [Numerical Analysis community]

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Task [Solve Linear time-invariant (LTI) State-Space Systems]:

$$\boldsymbol{M} \cdot \partial_t \vec{x} = \boldsymbol{A} \cdot \vec{x} + \boldsymbol{B} \cdot \vec{c} \qquad \text{[ODE system]}$$
$$\vec{y} = \boldsymbol{C} \cdot \vec{x} \qquad \text{[Observation relation]}$$

where $\vec{\mathrm{x}} \in \mathbb{R}^N$ is the system state, $\vec{u} \in \mathbb{R}^M$ is the controllable system input, and $\vec{y} \in \mathbb{R}^P$ is the observable system output. The system matrices are thus of dimension

$$\mathbf{M}, \mathbf{A} \in \mathbb{R}^{N\times N}, \qquad \mathbf{B} \in \mathbb{R}^{N\times M}, \qquad \text{and } \mathbf{C} \in \mathbb{R}^{P\times N}$$

with typically $M \ll \mathrm{N}$ and $P \ll N$.

- For simplicity let us ignore in the following $\vec{y}$, respectively assume $\vec{\mathrm{y}} = \vec{\mathrm{x}}$ or $\boldsymbol{C} = \boldsymbol{I}$
- Krylov subspace model order reduction methods have emerged in the field of large electronic circuit as well as micro-electro-mechanical systems (MEMS). The goal is to reduce systems with many degrees of freedoms to systems with less degrees of freedom but a similar input output behavior.
- Main purpose of Krylov methods is the construction of an appropriate system transfer function which accurately enough describes the dependence between the input and the output of the original system

Reduced Order Modelling
Refresher: Laplace Transform

- The *Laplace transform* is an integral transform that converts a function of a real variable in the time domain to a function of a complex variable in the frequency domain (s-domain):

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t)\exp(-st)\,dt$$

- Its inverse is given by

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT}\exp(st)F(s)\,ds\,,$$

where γ is an appropriate real number.

- The Laplace transform converts differentiation and integration in the time domain into much easier multiplication and division in the Laplace domain, e.g.[1],

$\boldsymbol{M}\cdot\partial_t\vec{x} = \boldsymbol{A}\cdot\vec{x} + \boldsymbol{B}\cdot\vec{c}$	$\Leftrightarrow$	$s\boldsymbol{M}\cdot\vec{x} = \boldsymbol{A}\cdot\vec{x} + \boldsymbol{B}\cdot\vec{c}$
Time domain		***Frequency domain***

1) For convenience we do not differentiate the notation between the time and frequency domain

Reduced Order Modelling

Krylov Subspace Model Order Reduction (II)

- Performing a *Laplace Transform* on the system yields

$$s\boldsymbol{M}\cdot\vec{x} = \boldsymbol{A}\cdot\vec{x} + \boldsymbol{B}\cdot\vec{c}$$

respectively $\vec{x}(s) = (s\boldsymbol{M} - \boldsymbol{A})^{-1}\cdot\boldsymbol{B}\cdot\vec{c}$ with transfer function $\boldsymbol{G}(s) \coloneqq (s\boldsymbol{M} - \boldsymbol{A})^{-1}\boldsymbol{B}$.

- Let us now reformulate $(s\boldsymbol{M} - \boldsymbol{A})^{-1}$ as follows

$$(s\boldsymbol{M} - \boldsymbol{A})^{-1} = (s\boldsymbol{A}\cdot\boldsymbol{A}^{-1}\cdot\boldsymbol{M} - \boldsymbol{A})^{-1} = -(\mathbf{I} - s\boldsymbol{A}^{-1}\cdot\boldsymbol{M})^{-1}\cdot\boldsymbol{A}^{-1}$$

$$= -\sum_{j=0}^{\infty} s^j(\boldsymbol{A}^{-1}\cdot\boldsymbol{M})^j\cdot\boldsymbol{A}^{-1}$$

Neumann Series:
$(\mathbf{1} - \boldsymbol{A})^{-1} = \sum_{j=0}^{\infty}\boldsymbol{A}^j$

- $\vec{x}(s) = (s\boldsymbol{M} - \boldsymbol{A})^{-1}\cdot\boldsymbol{B}\cdot\vec{c}$

$$= -\sum_{j=0}^{\infty} s^j(\boldsymbol{A}^{-1}\cdot\boldsymbol{M})^j\cdot\boldsymbol{A}^{-1}\cdot\boldsymbol{B}\cdot\vec{c} = -\sum_{j=0}^{\infty}\boldsymbol{m}^j\cdot s^j\vec{c},$$

i.e., a Tayler expansion of in $\vec{x}(s)$ in s with moments $\boldsymbol{m}^j = (\boldsymbol{A}^{-1}\cdot\boldsymbol{M})^j\cdot A^1\cdot B \in \mathbb{R}^{N\times M}$

- Note: $\vec{x}(s)$ is given by an ∞-dimensional basis $\left\{\boldsymbol{m}^j\cdot\vec{c}\right\}_{j=0}^{\infty}$ and coefficients $-s^j$

Reduced Order Modelling
Krylov Subspace Model Order Reduction (III)

- Instead of working with an infinite Talor series, let's adopt a Taylor Approximation of order k, i.e.,

$$\vec{x}(s) \approx \vec{x}_a(s) := -\sum_{j=0}^{k} \boldsymbol{m}^j \cdot s^j \vec{c}$$

and thus

$$\vec{x}_a(s) \in \operatorname{span}\left\{\boldsymbol{m}^0 \cdot \vec{c}, \dots, \boldsymbol{m}^j \cdot \vec{c}\right\}$$

$$\begin{aligned}
&= \operatorname{span}\left\{\boldsymbol{m}^0, \dots, \boldsymbol{m}^j\right\} \cdot \vec{c} \\
&= \operatorname{span}\left\{\boldsymbol{A}^{-1} \cdot \boldsymbol{B}, \boldsymbol{A}^{-1} \cdot \boldsymbol{M} \cdot (\boldsymbol{A}^{-1} \cdot \boldsymbol{B}), \dots, (\boldsymbol{A}^{-1} \cdot \boldsymbol{M})^j \cdot (\boldsymbol{A}^{-1} \cdot \boldsymbol{B})\right\} \cdot \vec{c} \\
&= \mathbb{K}_r(\boldsymbol{A}^{-1} \cdot \boldsymbol{M}, \boldsymbol{A}^{-1} \cdot \boldsymbol{B}\,) \cdot \vec{c}
\end{aligned}$$

Definition [Krylov Subspace]:
For a matrix $\boldsymbol{M} \in \mathbb{R}^{N \times N}$ and a vector $\vec{v} \in \mathbb{R}^N$ the Krylov subspace or order $r \leq N$ is given by

$$\mathbb{K}_r(\boldsymbol{M}, \vec{v}) := \operatorname{span}\{\vec{v}, \boldsymbol{M} \cdot \vec{v}, \dots, \boldsymbol{M}^r \cdot \vec{v}\}$$

Reduced Order Modelling
Krylov Subspace Model Order Reduction (IV)

Idea [Moment Matching]:
The Krylov subspace $\mathbb{K}_r(\boldsymbol{A}^{-1} \cdot \boldsymbol{M}, \boldsymbol{A}^{-1} \cdot \boldsymbol{B})$ is spanned by the first r input moments. This leads to the idea to project the LTI system onto this subspace.

Orthogonal Projection:
Let v_1, ..., v_k be an orthonormal basis of $\mathbb{K}_r(\boldsymbol{A}^{-1} \cdot \boldsymbol{M}, \boldsymbol{A}^{-1} \cdot \boldsymbol{B})$ and define the matrix $\boldsymbol{V}$ column-wise by
$\boldsymbol{V} \coloneqq (v_1, \dots, v_k) \in \mathbb{R}^{N \times K}$.
This is an orthogonal projection onto the Krylov space by

$$\boldsymbol{V} \cdot \boldsymbol{V}^T \colon \mathbb{R}^N \to \mathbb{K}_r(\boldsymbol{A}^{-1} \cdot \boldsymbol{M}, \boldsymbol{A}^{-1} \cdot \boldsymbol{B})$$

with $\boldsymbol{V} \cdot \boldsymbol{V}^T = \boldsymbol{I}$ on $\mathbb{K}_r(\boldsymbol{A}^{-1} \cdot \boldsymbol{M}, \boldsymbol{A}^{-1} \cdot \boldsymbol{B})$

- Any $x_a \in \mathbb{K}_r(\boldsymbol{A}^{-1} \cdot \boldsymbol{M}, \boldsymbol{A}^{-1} \cdot \boldsymbol{B})$ can be represented by a r-dimensional coefficient vector $\vec{w}_r \coloneqq \boldsymbol{V}^T \cdot \vec{x}_a$with $\boldsymbol{V} \cdot \vec{w}_r \coloneqq \boldsymbol{V} \cdot \boldsymbol{V}^T \cdot \vec{x}_a = \vec{x}_a$.

Reduced Order Modelling
Krylov Subspace Model Order Reduction (V)

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Reduced Approximation:
For $\vec{x} \in \mathbb{R}^{\boldsymbol{N}}$ we have $\vec{x} = \boldsymbol{V} \cdot \boldsymbol{V}^T \cdot \vec{\boldsymbol{x}} + (\boldsymbol{I} - \boldsymbol{V} \cdot \boldsymbol{V}^T) \cdot \vec{\boldsymbol{x}}$ and the r-dimensional representation in the Krylov subspace $\mathbb{K}_r(\boldsymbol{A}^{-1} \cdot \boldsymbol{M}, \boldsymbol{A}^{-1} \cdot \boldsymbol{B})$ is not exact since $(\boldsymbol{I} - \boldsymbol{V} \cdot \boldsymbol{V}^T)$ is missing. That is,
$\vec{x}_r := \boldsymbol{V}^T \cdot \vec{x} \in \mathbb{R}^r \;\Rightarrow\; \boldsymbol{V} \cdot \vec{x}_r = \boldsymbol{V} \cdot \boldsymbol{V}^T \cdot \vec{x} \approx \vec{x}$

Reduced LTI System:
Inserting the reduced approximation of $\vec{x} \approx \boldsymbol{V} \cdot \vec{x}_r$in the LTI system and multiplying with $\boldsymbol{V}^T$ from the left, gives the reduced LTI system, i.e.,

$$\boldsymbol{V}^T \cdot \boldsymbol{M} \cdot \boldsymbol{V} \cdot \partial_t \vec{x}_r = \boldsymbol{V}^T \cdot \boldsymbol{A} \cdot \boldsymbol{V} \cdot \vec{x}_r + \boldsymbol{V}^T \cdot \boldsymbol{B} \cdot \vec{c} \qquad \text{[ODE system]}$$

$$\vec{\boldsymbol{y}} = \boldsymbol{C} \cdot \boldsymbol{V} \cdot \vec{x}_r \qquad \text{[Observation relation]}$$

Main Theorem:
For the transfer function $\boldsymbol{G}_r(s) := (s\boldsymbol{M}_r - \boldsymbol{A}_r)^{-1}\boldsymbol{B}_r$ corresponding to the reduced LTI system and the transfer function $\boldsymbol{G}(s) := (s\boldsymbol{M} - \boldsymbol{A})^{-1}\boldsymbol{B}$ corresponding to the initial full system, the first r moments match.

Reduced Order Modelling
Krylov Subspace Model Order Reduction (VI)

Reduced LTI System:

Inserting the reduced approximation of $\vec{x} \approx \boldsymbol{V} \cdot \vec{x}_r$ in the LTI system and multiplying with $\boldsymbol{V}^T$ from the left, gives the reduced LTI system, i.e.,

$$\boldsymbol{V}^T \cdot \boldsymbol{M} \cdot \boldsymbol{V} \cdot \partial_t \vec{x}_r = \boldsymbol{V}^T \cdot \boldsymbol{A} \cdot \boldsymbol{V} \cdot \vec{x}_r + \boldsymbol{V}^T \cdot \boldsymbol{B} \cdot \vec{c} \qquad \text{[ODE system]}$$

$$\vec{\boldsymbol{y}} = \boldsymbol{C} \cdot \boldsymbol{V} \cdot \vec{x}_r \qquad \text{[Observation relation]}$$

- We have

$$\begin{array}{r} \vec{x} \in \mathbb{R}^N \\ \mathbf{M}, \mathbf{A} \in \mathbb{R}^{N\times N} \\ \mathbf{B} \in \mathbb{R}^{N\times M} \\ \mathbf{C} \in \mathbb{R}^{P\times N} \end{array} \xrightarrow[\boldsymbol{V}^T \in \mathbb{R}^{r\times N}]{\boldsymbol{V} \in \mathbb{R}^{N\times r}} \begin{array}{r} \vec{x}_r \in \mathbb{R}^r \\ \boldsymbol{V}^T \cdot \mathbf{M} \cdot \boldsymbol{V}, \boldsymbol{V}^T \cdot \boldsymbol{A} \cdot \boldsymbol{V} \in \mathbb{R}^{r\times r} \\ \boldsymbol{V}^T \cdot \mathbf{B} \in \mathbb{R}^{r\times M} \\ \mathbf{C} \cdot \boldsymbol{V} \in \mathbb{R}^{P\times r} \end{array}$$

and thus a reduced LTI system of a much smaller dimension, which can be solved very efficiently.

Reduced Order Modelling
Krylov Subspace Model Order Reduction (VII)

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- But what is a good basis for a Krylov space $\mathbb{K}(\boldsymbol{M}, \vec{b}) = \text{span}\{\vec{b}, \boldsymbol{M} \cdot \vec{b}, \dots, \boldsymbol{M}^r \cdot \vec{b}\}$?

Algorithm [Arnoldi]:

Set $H_{1,1} := \|\vec{b}\|$ and $v_1 := H_{1,1}^{-1}\vec{b}$

For $k = 1, \dots, t-1$

$\quad \vec{w}_{k-1} := \boldsymbol{M} \cdot \vec{v}_k$ [Apply next power of $\boldsymbol{M}$]

$\quad$ For $l = 1, \dots, k$ [Orthogonalization]

$\quad\quad H_{l,k} := \vec{v}_l \cdot \vec{v}_k$

$\quad\quad \vec{w}_{k+1} = \vec{w}_{k+1} - H_{l,k}\vec{v}_l$

$\quad H_{k+1,k} := \|\vec{w}_{k+1}\|$ [Compute Norm]

$\quad \vec{v}_{k+1} := H_{k+1,k}^{-1}\vec{w}_{k+1}$ [Normalize]

- To build the ONB for the Krylov subspace $\mathbb{K}_r(\boldsymbol{A}^{-1} \cdot \boldsymbol{M}, \boldsymbol{A}^{-1} \cdot \boldsymbol{B})$ apply the Arnoldi algorithm to $\boldsymbol{M} := \boldsymbol{A}^{-1} \cdot \boldsymbol{M}$ and "$\vec{b}$" $:= \boldsymbol{A}^{-1} \cdot \boldsymbol{B}$.

- In general "$\vec{\text{b}}$" $:= \boldsymbol{A}^{-1} \cdot \boldsymbol{B} \in \mathbb{R}^{N \times M}$ is a matrix. This leads to the so-called Block-Arnoldi iteration.

Reduced Order Modelling
Krylov Subspace Model Order Reduction

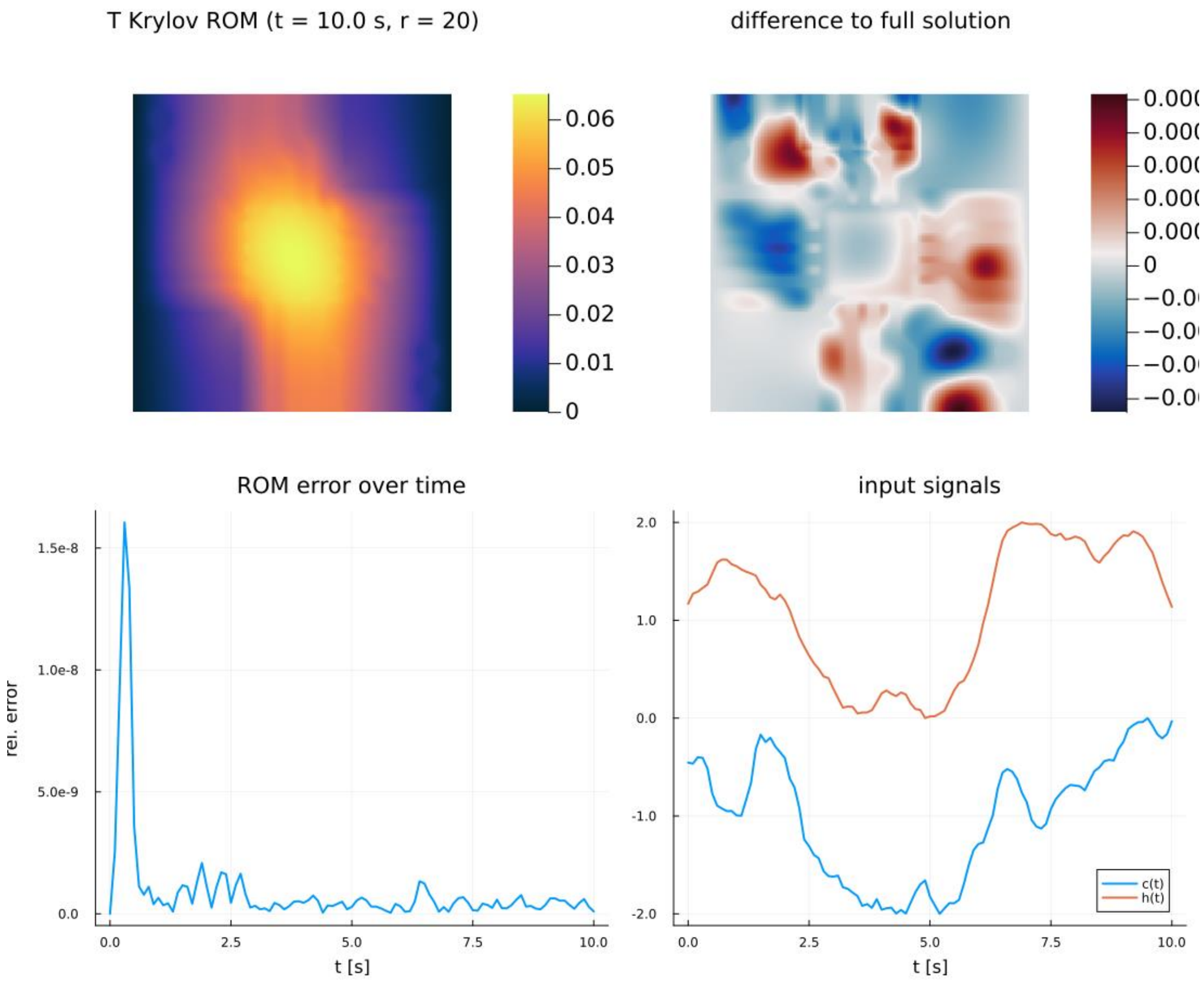
- Krylov Subspace Model Order Reduction allows to reduce the complexity of an LTI system effectively. Since r is in general much smaller than N, computationally very efficient models can be built.

- Extensions to the wave equation and other linear systems are possible.

- Krylov Subspace Model Order Reduction (As many other similar methods) relies heavily on the structure of the LTI system. That is, the matrix $\boldsymbol{A}$ is constant, i.e., no parameterizations or changes in geometry are possible.

- Since each basis vector of the Krylov Basis has a representation in $\mathbb{R}^N$, we can consider the resulting system as a discretization obtained by a corresponding spectral basis.

Reduced Order Modelling
Krylov Subspace Model Order Reduction



https://github.com/Dirk-Hartmann/real-time-digital-twins/blob/main/scripts/rom/PCB_KrylovROM.jl

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Reduced Order Modelling

Balanced Truncation [System and Control community]

- Originated from the field of dynamical systems and design of controllers for these systems, which are heart to many engineering problems.

- Also works in the frequency space and focuses on an appropriate approximation of the transfer function using rational interpolation.

- Core focus is to preserve the stability properties of the high-order model and to provide an error bound which gives a direct measure of the quality of the reduced order model. Thus, the reduced order model can be generated automatically.

Reduced Order Modelling

Comparison of intrusive projection-based methods (i)

	Mode Displacement	Moment Matching / Krylov	Balanced Truncation*
Origin	Mechanical systems	Electric systems and numerical analysis	System and control Community
Reduction Method	Intrusive projection-based	Intrusive projection-based	Intrusive projection-based
Addressed Systems	Second-order systems	First-order (LTI) systems[1]	First-order (LTI) systems[1]
Reduction Focus	Global behavior, e.g., regions of high stress	Input-output behavior	Input-output behavior
Reduction Mechanism	Frequency-based with limited coverage of frequency domain (Defined by selected eigen modes)	Frequency-based with good approximation around the expansion point of the Taylor series	Energy-based focus and thus more a time-domain-based approach

*Not covered in this course

1) Thus, also higher-order systems if converted to a first-order system

Source: B Besselink, U Tabak, A Lutowska, N van de Wouw, H Nijmeijer, DJ Rixen, M Hochstenbach, W Schilders (2013): A comparison of model reduction techniques from structural dynamics, numerical mathematics and systems and control. Journal of Sound and Vibration

Reduced Order Modelling
Comparison of intrusive projection-based methods (ii)

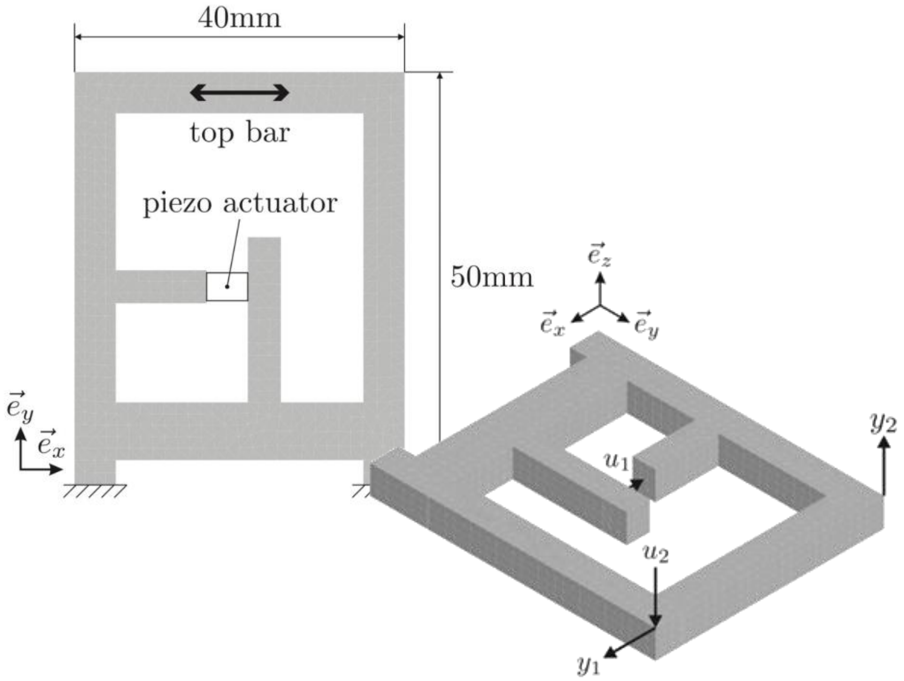
	Displacement Method	Moment Matching / Krylov	Balanced Truncation*
Physical Interpretability	yes	no	no
Stability Preservation	Limited	Limited	Preserved
Automation	Limited	Limited	Yes, due to error bounds.
Computational Costs	Medium, due to availability of effective eigen solvers	Medium, due to availability of effective eigen solvers	High

*Not covered in this course

Source: B Besselink, U Tabak, A Lutowska, N van de Wouw, H Nijmeijer, DJ Rixen, M Hochstenbach, W Schilders (2013): A comparison of model reduction techniques from structural dynamics, numerical mathematics and systems and control. Journal of Sound and Vibration

Model Order Reduction
Example (i)

- Structural analysis example: Steel frame which is driven by a piezo-electric actuator to control the displacement in the $\vec{e}_x$ direction of the top bar. After discretization with the finite element method the following system with $N = 8730$ is obtained:



$$M \cdot \partial_{tt}\vec{q} + D \cdot \partial_t\vec{q} + K \cdot \vec{q} = \vec{b}_1 u_1 + \vec{b}_2 u_2$$

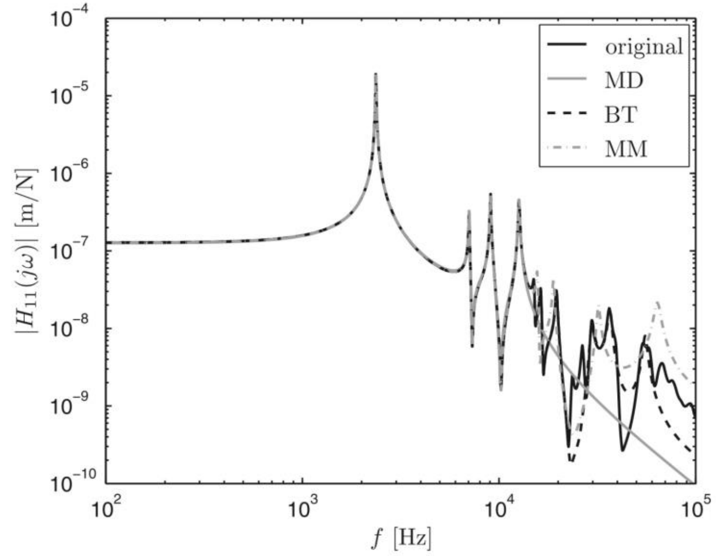
$$y_1 = \vec{c}_1^{\,t} \cdot \vec{q}, y_2 = \vec{c}_2^{\,t} \cdot \vec{q}$$

$\vec{q}$ is the vector of displacements of nodes, $M$is the mass matrix, K the stiffness matric. u_1 modls the actuator force (with generalized force directions b_1) and u_2 a disturbance in the $e_z$direction (with generalized force directions b_1). y_1 and y_2 measure the displacements of some corner points of the frame

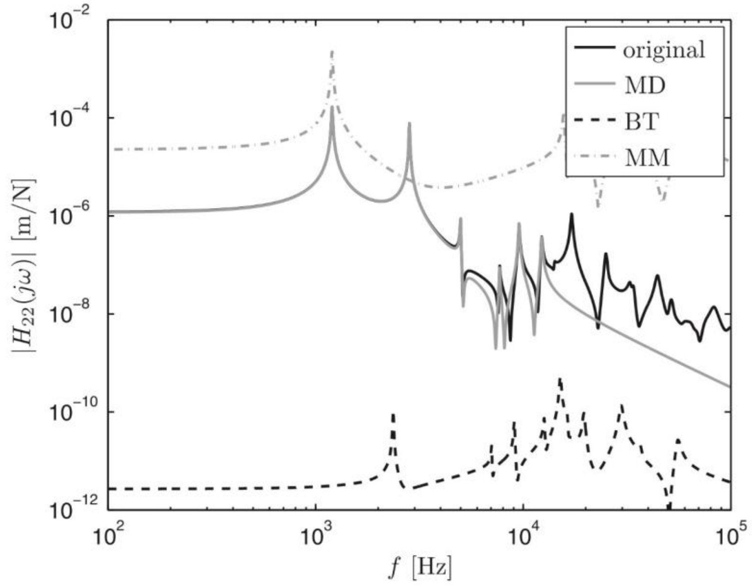
Source: B Besselink, U Tabak, A Lutowska, N van de Wouw, H Nijmeijer, DJ Rixen, M Hochstenbach, W Schilders (2013): A comparison of model reduction techniques from structural dynamics, numerical mathematics and systems and control. Journal of Sound and Vibration

Model Order Reduction
Example (ii)

- Comparison of the mode displacement method (MD), balanced truncation (BT) and moment matching (MM)/ Krylov for reduction to 10 for MD respectively 20 modes for BT and MM.
- For BT and MM the ROM is based on input u_1 and output y_1



Magnitude of the frequency response function for input u_1 and output y_1



Magnitude of the frequency response function for input u_2 and output y_2

Source: B Besselink, U Tabak, A Lutowska, N van de Wouw, H Nijmeijer, DJ Rixen, M Hochstenbach, W Schilders (2013): A comparison of model reduction techniques from structural dynamics, numerical mathematics and systems and control. Journal of Sound and Vibration

Model Order Reduction
Wrap Up

- Focus of most Model Order Reduction Techniques is Linear Time Invariant Systems

- (Limited) extensions to parameterized and non-linear systems are available

- Drawback of all methods: We need access to the matrices in order to reconstruct the models

- **Opportunity:** Data-driven Model Order Reduction

  Examples include:
  - *System Identification* [frequency and time domain]
  - *Neural Networks* [time domain]
  - *Lowner interpolation* [frequency domain]
  - *Koopman / Dynamic Mode Decomposition* (DMD) [Time domain]
  - *Operator Inference* [Time Domain]

Contact

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